

Cosmic Inflation

A. Expansion of Universe

Question A.1

Answer	Marks
For any test mass m on the boundary of the sphere, $m\ddot{R}(t) = -GmM_s/R^2(t) \quad (\text{A.1.1})$ where M_s is mass portion inside the sphere	0.2
Multiplying equation (A.1.1) with $\dot{R}$ and integrating it gives $\int \dot{R} \frac{d\dot{R}}{dt} dt = \frac{1}{2} \dot{R}^2 = \frac{GM_s}{R} + A$ where A is a integration constant	0.6
Taking $M_s = \frac{4}{3}\pi R^3(t)\rho(t)$, and $\dot{R} = \dot{a} R_s$	0.2
$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3} \rho(t) + \frac{2A}{R_s^2 a^2(t)}$	0.2
Therefore, we have $A_1 = \frac{8\pi G}{3}$	0.1
Total	1.3

Question A.2

Answer	Marks
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Solutions/ Marking Scheme



T3

The 2 nd Friedmann equation can be obtained from the 1 st law of thermodynamics :	0.1
$dE = -pdV + dQ.$	
For adiabatic processes $dE + pdV = 0$ and its time derivative is $\dot{E} + p \dot{V} = 0$.	0.1
For the sphere $\dot{V} = V (3 \dot{a}/a)$	0.1
Its total energy is $E = \rho(t)V(t) c^2$	0.2
Therefore $\dot{E} = \left(\dot{\rho} + 3 \frac{\dot{a}}{a}\right) V c^2$	0.1
It yields	0.2
$\dot{\rho} + 3 \left(\rho + \frac{p}{c^2}\right) \frac{\dot{a}}{a} = 0$	
Therefore, we have $A_2 = 3$.	0.1
Total	0.9

Question A.3

Answer	Marks
Interpreting $\rho(t)c^2$ as total energy density, and substituting $\frac{p(t)}{c^2} = w \rho(t)$ in to the 2 nd Friedmann equation yields: $\dot{\rho} + 3 \rho(1 + w) \frac{\dot{a}}{a} = 0$	0.1
$\rho \propto a^{-3(w+1)}$	0.2
(i) In case of radiation, photon as example, the energy is given by $E_r = h\nu = hc/\lambda$ then its energy density $\rho_r = \frac{E_r}{V} \propto a^{-4}$ so that $w_r = \frac{1}{3}$	0.3
(ii) In case of nonrelativistic matter, its energy density nearly $\rho_m \simeq \frac{m_0 c^2}{V} \propto a^{-3}$ since dominant energy comes from its rest energy $m_0 c^2$, so that $w_m = 0$	0.3
(iii) For a constant energy density, let say $\epsilon_\Lambda = \text{constant}$, $\epsilon_\Lambda \propto a^0$ so that $w_\Lambda = -1$.	0.3
Total	1.2

Question A.4

Answer	Marks
(i) In case of $k = 0$, for radiation we have $\rho_r a^4 = \text{constant}$. So by comparing the parameters values with their present value, $\rho_r(t) a^4(t) = \rho_{r0} a_0^4$, $\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3} \rho_{r0} \left(\frac{a_0}{a}\right)^4.$ $\int a da = \frac{1}{2} a^2 + K = \left(\frac{8\pi G}{3} \rho_{r0} a_0^4\right)^{\frac{1}{2}} t.$	0.2
Because $a(t = 0) = 0, K = 0$, then $a(t) = (2)^{\frac{1}{2}} \left(\frac{8\pi G}{3} \rho_{r0} a_0^4\right)^{\frac{1}{4}} t^{\frac{1}{2}} = (2H_0)^{\frac{1}{2}} t^{\frac{1}{2}}.$ where $H_0 = \left(\frac{8\pi G}{3} \rho_{r0}\right)^{\frac{1}{2}}$ after taking $a_0 = 1$.	0.2
(ii) for non-relativistic matter domination, using $\rho_m(t) a^3(t) = \rho_{m0} a_0^3$, and similar way we will get $a(t) = \left(\frac{3}{2}\right)^{\frac{2}{3}} \left(\frac{8\pi G}{3} \rho_{m0} a_0^4\right)^{\frac{1}{3}} t^{\frac{2}{3}} = \left(\frac{3H_0}{2}\right)^{\frac{2}{3}} t^{\frac{2}{3}}.$ where $H_0 = \left(\frac{8\pi G}{3} \rho_{m0}\right)^{\frac{1}{2}}$.	0.4
(iii) for constant energy density, $\ln a = H_0 t + K'$ Where K' is integration constant and $H_0 = \left(\frac{8\pi G}{3} \rho_\Lambda\right)^{\frac{1}{2}}$. Taking condition $a_0 = 1$, $\ln\left(\frac{a}{a_0}\right) = H_0(t - t_0)$ $a(t) = e^{H_0(t-t_0)}$	0.4
Total	1.2

Question A.5

Answer	Marks
Condition for critical energy condition: $\rho_c(t) = \frac{3H^2}{8\pi G}$ Friedmann equation can be written as $H^2(t) = H^2(t)\Omega(t) - \frac{kc^2}{R_0^2 a^2(t)}$ $\left(\frac{R_0^2}{c^2}\right) a^2 H^2 (\Omega - 1) = k \quad (\text{A.5.1})$	0.1
Total	0.1

Question A.6

Answer	Marks
Because $\left(\frac{R_0^2}{c^2}\right) a^2 H^2 > 0$, then $k = +1$ corresponds to $\Omega > 1$, $k = -1$ corresponds to $\Omega < 1$ and $k = 0$ corresponds to $\Omega = 1$	0.3
Total	0.3

B. Motivation To Introduce Inflation Phase and Its General Conditions

Question B.1

Answer	Marks
Equation (A.5.1) shows that $(\Omega - 1) = \frac{kc^2}{R_0^2} \frac{1}{\dot{a}^2}.$	0.1
In a universe dominated by non-relativistic matter or radiation, scale factor can be written as a function of time as $a = a_0 \left(\frac{t}{t_0}\right)^p$ where $p < 1$ ($p = \frac{1}{2}$ for radiation and $p = \frac{2}{3}$ for non-relativistic matter)	0.2
$(\Omega - 1) = \tilde{k} t^{2(1-p)}$	0.2
Total	0.5

Question B.2

Answer	Marks
For a period dominated by constant energy provides the solution $a(t) = e^{Ht}$ so that $\dot{a} = He^{Ht}$	0.1
$(\Omega - 1) = \frac{k}{H^2} t^{-2Ht}$	0.2
Total	0.3

Question B.3

Answer	Marks
Inflation period can be generated by constant energy period, therefore it is a phase where $w = -1$ so that $p = w\rho c^2 = -\rho c^2$ (negative pressure).	0.2
Differentiating Friedmann equation leads to $\dot{a}^2 = \frac{8\pi G}{3} \rho a^2 - \frac{kc^2}{R_0^2}$ $2\dot{a}\ddot{a} = \frac{8\pi G}{3} (\dot{\rho}a^2 + 2\rho a \dot{a}) = \frac{8\pi G}{3} (-3 \left(\rho + \frac{p}{c^2}\right) a\dot{a} + 2\rho a\dot{a}).$ $\frac{\ddot{a}}{a} = -\frac{4\pi G}{3} \left(\rho + \frac{3p}{c^2}\right)$	0.4
So that because during inflation $p = -\rho c^2$, it is equivalent with condition $\ddot{a} > 0$ (accelerated expansion)	0.1
As a result, $\ddot{a} = d(\dot{a})/dt = d(Ha)/dt > 0$ or $d(Ha)^{-1}/dt < 0$ (shrinking Hubble radius).	0.2
Total	0.9

Question B.4

Answer	Marks
Inflation condition can be written as $\frac{d(aH)^{-1}}{dt} < 0$, with $H = \dot{a}/a$ as such $\frac{d(aH)^{-1}}{dt} = -\frac{\dot{a}H + a\dot{H}}{(aH)^2} = -\frac{1}{a}(1 - \epsilon) < 0 \Rightarrow \epsilon < 1$	0.2
Total	0.2

C. Inflation Generated by Homogenously Distributed Matter

Question C.1

Answer	Marks
Differentiating equations (4) and employing equation 4 we can get $2H\dot{H} = \frac{1}{3M_{pl}^2} \left[\dot{\phi}\ddot{\phi} + \left(\frac{\partial V}{\partial \phi} \right) \dot{\phi} \right] = \frac{1}{3M_{pl}^2} [-3H\dot{\phi}^2]$ $\dot{H} = -\frac{1}{2} \frac{\dot{\phi}^2}{M_{pl}^2}$	0.3
Therefore $\epsilon = \frac{1}{2} \frac{\dot{\phi}^2}{M_{pl}^2 H^2}$	0.1
The inflation can occur when the potential energy dominates the particle's energy ($\dot{\phi}^2 \ll V$) such that $H^2 \approx V/(3M_{pl}^2)$.	0.2
Slow-roll approximation: $3H\dot{\phi} \approx -V'$	0.1
Implies $\epsilon \approx \frac{M_{pl}^2}{2} \left(\frac{V'}{V} \right)^2 \quad (C.1.1)$	0.3
we also have $3\dot{H}\dot{\phi} + 3H\ddot{\phi} = -V''\dot{\phi}$ $\delta = -\frac{\ddot{\phi}}{H\dot{\phi}} = \frac{V''}{3H^2} - \epsilon$ Therefore $\eta_V \approx M_{Pl}^2 \frac{V''}{V} \quad (C.1.2)$	0.4
$dN = H dt = \left(\frac{H}{\dot{\phi}} \right) d\phi \approx -\frac{1}{M_{pl}^2} (V/V') d\phi \quad (C.1.3)$ $\frac{dN}{d\phi} \approx -\frac{1}{M_{pl}^2} (V/V')$	0.3
Total	1.7

D. Inflation with A Simple Potential

Question D.1

Answer	Marks
Inflation ends at $\epsilon = 1$. Using $V(\phi) = \Lambda^4 (\phi/M_{pl})^n$ yields $\epsilon = \frac{M_{pl}^2}{2} \left[\frac{n}{\phi_{end}} \right]^2 = 1 \Rightarrow \phi_{end} = \frac{n}{\sqrt{2}} M_{pl}$	0.5
Total	0.5

Question D.2

Answer	Marks
From equations (C.1.1), (C.1.2) and (C.1.3) we can obtain $N = - \left[\frac{\phi}{M_{pl}} \right]^2 \frac{1}{2n} + \beta$ where β is a integration constant. As $N = 0$ at ϕ_{end} then $\beta = \frac{n}{4}$. $N = - \left[\frac{\phi}{M_{pl}} \right]^2 \frac{1}{2n} + \frac{n}{4}$	0.2
$\eta_V = n(n-1) \left[\frac{M_{pl}}{\phi} \right]^2 = \frac{2(n-1)}{n-4N}$	0.2
$\epsilon = \frac{n^2}{2} \left[\frac{M_{pl}}{\phi} \right]^2 = \frac{n}{n-4N}$	0.2
so that $r = 16\epsilon = \frac{16n}{n-4N}$	0.1

Solutions/ Marking Scheme



T3

$n_s = 1 + 2\eta_V - 6\epsilon = 1 - \frac{2(n+2)}{(n-4N)}$	0.1
To obtain the observational constraint $n_s = 0.968$ we need $n = -5.93$ which is inconsistent with the condition $r < 0.12$. There is <u>no a closest integer</u> n that can obtains $r < 0.12$. As example, for $n = -6$ leads a contradiction $0 < (-0.27)$ and for $n = -5$ leads a contradiction $0 < (-0.2)$.	0.1
Total	0.9