

# The Extremum Principle Solution

## A. The Extremum Principle in Mechanics

**Deduct 0.1 if units missing in final answer upto a maximum of 0.5 points**

Consider a horizontal frictionless  $x$ - $y$  plane shown in Fig. 1. It is divided into two regions, I and II, by a line AB satisfying the equation  $x = x_1$ . The potential energy of a point particle of mass  $m$  in region I is  $V = 0$  while it is  $V = V_0$  in region II. The particle is sent from the origin O with speed  $v_1$  along a line making an angle  $\theta_1$  with the  $x$ -axis. It reaches point P in region II traveling with speed  $v_2$  along a line that makes an angle  $\theta_2$  with the  $x$ -axis. Ignore gravity and relativistic effects in this entire task T-2 (all parts).

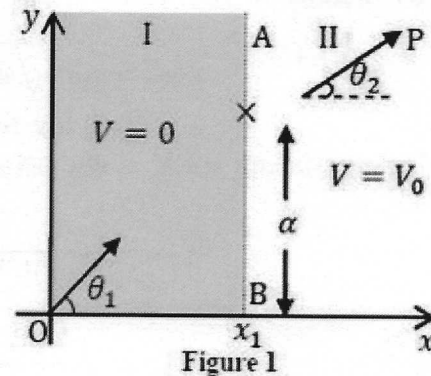


Figure 1

(A1) Obtain an expression for  $v_2$  in terms of  $m$ ,  $v_1$  and  $V_0$ .

[0.2]

**Solution:**

From the principle of Conservation of Mechanical Energy

$$\frac{1}{2}mv_1^2 = \frac{1}{2}mv_2^2 + V_0$$

0.1

$$v_2 = (v_1^2 - \frac{2V_0}{m})^{1/2}$$

0.1

(A2) Express  $v_2$  in terms of  $v_1$ ,  $\theta_1$  and  $\theta_2$ .

[0.3]

**Solution:**

At the boundary there is an impulsive force ( $\propto dV/dx$ ) in the  $-x$  direction. Hence only the velocity component in the  $x$ -direction  $v_{1x}$  suffers change. The component in the  $y$ -direction remains unchanged. Therefore

$$v_{1y} = v_{2y}$$

$$v_1 \sin \theta_1 = v_2 \sin \theta_2$$

0.2

0.1

We define a quantity called action  $A = m \int v(s) ds$ , where  $ds$  is the infinitesimal length along the trajectory of a particle of mass  $m$  moving with speed  $v(s)$ . The integral is taken over the path. As an example, for a particle moving with constant speed  $v$  on a circular path of radius  $R$ , the action  $A$  for one revolution will be  $2\pi m R v$ . For a particle with constant energy  $E$ , it can be shown that of all the possible trajectories between two fixed points, the actual trajectory is the one on which  $A$  defined above is an extremum (minimum or maximum). Historically this is known as the Principle of Least Action (PLA).

- (A3) PLA implies that the trajectory of a particle moving between two fixed points in a region of constant potential will be a straight line. Let the two fixed points O and P in Fig. 1 have coordinates (0,0) and  $(x_0, y_0)$  respectively and the boundary point where the particle transits from region I to region II have coordinates  $(x_1, \alpha)$ . Note  $x_1$  is fixed and the action depends on the coordinate  $\alpha$  only. State the expression for the action  $A(\alpha)$ . Use PLA to obtain the relationship between  $v_1/v_2$  and these coordinates. [1.0]

**Solution:**

By definition  $A(\alpha)$  from O to P is

$$A(\alpha) = mv_1 \sqrt{x_1^2 + \alpha^2} + mv_2 \sqrt{(x_0 - x_1)^2 + (y_0 - \alpha)^2} \quad 0.3$$

(0.1 for first and 0.2 for second expression = 0.3 points)

Differentiating w.r.t.  $\alpha$  and setting the derivative of  $A(\alpha)$  to zero 0.3

$$\frac{v_1 \alpha}{(x_1^2 + \alpha^2)^{1/2}} - \frac{v_2 (y_0 - \alpha)}{[(x_0 - x_1)^2 + (y_0 - \alpha)^2]^{1/2}} = 0 \quad 0.2$$

$$\therefore \frac{v_1}{v_2} = \frac{(y_0 - \alpha) (x_1^2 + \alpha^2)^{1/2}}{\alpha [(x_0 - x_1)^2 + (y_0 - \alpha)^2]^{1/2}} \quad 0.2$$

Note this is the same as A2, namely  $v_1 \sin \theta_1 = v_2 \sin \theta_2$ .

## B. The Extremum Principle in Optics

A light ray travels from medium I to medium II with refractive indices  $n_1$  and  $n_2$  respectively. The two media are separated by a line parallel to the x-axis. The light ray makes an angle  $i_1$  with the y-axis in medium I and  $i_2$  in medium II (see Fig. 2). To obtain the trajectory of the ray, we make use of another extremum (minimum or maximum) principle known as Fermat's principle of least time.

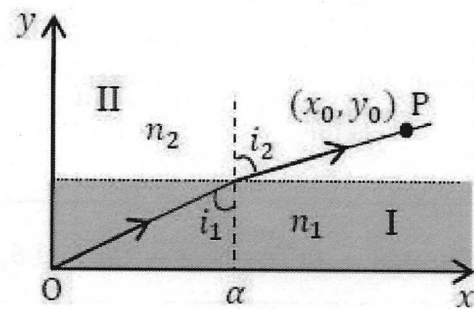


Figure 2

Figure 2

- (B1) The principle states that between two fixed points, a light ray moves along a path such that the time taken between the two points is an extremum. Derive the relation between  $\sin i_1$  and  $\sin i_2$  on the basis of Fermat's principle. [0.5]

**Solution:**

The speed of light in medium I is  $c/n_1$  and in medium II is  $c/n_2$ , where  $c$  is the speed of light in vacuum. Let the two media be separated by the fixed line  $y = y_1$ . Then time  $T(\alpha)$  for light to travel from origin (0,0) and  $(x_0, y_0)$  is

$$T(\alpha) = n_1 (\sqrt{y_1^2 + \alpha^2})/c + n_2 (\sqrt{(x_0 - \alpha)^2 + (y_0 - y_1)^2})/c \quad 0.2$$

(0.1 for first and 0.1 for second expression = 0.2 points)

Differentiating w.r.t.  $\alpha$  and setting  $T(\alpha)$  to zero

$$\frac{n_1 \alpha}{(y_1^2 + \alpha^2)^{1/2}} - \frac{n_2 (y_0 - \alpha)}{[(x_0 - \alpha)^2 + (y_0 - y_1)^2]^{1/2}} = 0$$

0.2

$$\therefore n_1 \sin i_1 = n_2 \sin i_2$$

0.1

[Note: Derivation is similar to A3. This is Snell's law.]

Shown in Fig. 3 is a schematic sketch of the path of a laser beam incident horizontally on a solution of sugar in which the concentration of sugar decreases with height. As a consequence, the refractive index of the solution also decreases with height.

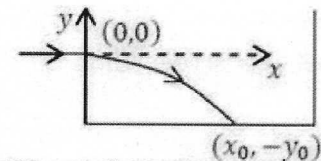


Figure 3: Tank of Sugar Solution

Figure 3

- (B2) Assume that the refractive index  $n(y)$  depends only on  $y$ . Use the equation obtained in B1 to obtain the expression for the slope  $dy/dx$  of the beam's path in terms of  $n_0$  and  $n(y)$ . [1.5]

**Solution:**

From Snell's law  $n_0 \sin i_0 = n(y) \sin i$

0.5

Then,  $\frac{dy}{dx} = -\cot i$

0.4

(0.2 marks for negative sign.)

Given  $n(y)$  and  $i_0 = 90^\circ$ , so  $\sin i_0 = 1$ .

$$n_0 = \frac{n(y)}{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}}$$

0.4

$$\frac{dy}{dx} = -\sqrt{\left(\frac{n(y)}{n_0}\right)^2 - 1}$$

0.2

- (B3) The laser beam is directed horizontally from the origin  $(0,0)$  into the sugar solution at a height  $y_0$  from the bottom of the tank as shown. Take  $n(y) = n_0 - ky$  where  $n_0$  and  $k$  are positive constants. Obtain an expression for  $x$  in terms of  $y$  and related quantities. You may use:  $\int \sec \theta d\theta = \ln(\sec \theta + \tan \theta) + \text{constant}$   $\sec \theta = 1/\cos \theta$  or  $\int \frac{dx}{\sqrt{x^2 - 1}} = \ln(x + \sqrt{x^2 - 1}) + \text{constant}$ . [1.2]

**Solution:**

$$\int \frac{dy}{\sqrt{\left(\frac{n_0 - ky}{n_0 \sin i_0}\right)^2 - 1}} = - \int dx$$

0.4

3

Note  $i_0 = 90^\circ$  so  $\sin i_0 = 1$ .

**Method I** We employ the substitution

$$\xi = \frac{n_0 - ky}{n_0}$$

$$\int \frac{d\xi \left(-\frac{n_0}{k}\right)}{\sqrt{\xi^2 - 1}} = - \int dx$$

Let  $\xi = \sec \theta$ . Then

$$\frac{n_0}{k} \ln(\sec \theta + \tan \theta) = x + c$$

**Or METHOD II**

We employ the substitution

$$\xi = \frac{n_0 - ky}{n_0}$$

$$\int \frac{d\xi \left(-\frac{n_0}{k}\right)}{\sqrt{\xi^2 - 1}} = - \int dx$$

$$-\frac{n_0}{k} \ln \left( \frac{n_0 - ky}{n_0} + \sqrt{\left(\frac{n_0 - ky}{n_0}\right)^2 - 1} \right) = -x + c$$

Considering the substitutions and boundary condition,  $x = 0$  for  $y = 0$  we obtain that the constant  $c = 0$ .

Hence we obtain the following trajectory:

$$x = \frac{n_0}{k} \ln \left( \frac{n_0 - ky}{n_0} + \sqrt{\left(\frac{n_0 - ky}{n_0}\right)^2 - 1} \right)$$

(B4) Obtain the value of  $x_0$ , the point where the beam meets the bottom of the tank. Take  $y_0 = 10.0$  cm,  $n_0 = 1.50$ ,  $k = 0.050$  cm<sup>-1</sup> (1 cm = 10<sup>-2</sup> m).

**Solution:**

Given  $y_0 = 10$  cm.       $n_0 = 1.5$        $k = 0.05$  cm<sup>-1</sup>  
From (C3)

$$x_0 = \frac{n_0}{k} \ln \left[ \left( \frac{n_0 - ky}{n_0} \right) + \left( \left( \frac{n_0 - ky}{n_0} \right)^2 - 1 \right)^{1/2} \right]$$

Here  $y = -y_0$

$$\begin{aligned}
 x_0 &= \frac{n_0}{k} \ln \left[ \frac{(n_0 + ky_0)}{n_0} + \left( \frac{(n_0 + ky_0)^2}{n_0^2} - 1 \right)^{1/2} \right] \\
 &= 30 \ln \left[ \frac{2}{1.5} + \left( \left( \frac{2}{1.5} \right)^2 - 1 \right)^{1/2} \right] \\
 &= 30 \ln \left[ \frac{4}{3} + \left( \frac{7}{9} \right)^{1/2} \right] \\
 &= 30 \ln \left[ \frac{4}{3} + 0.88 \right] \\
 &= 24.0 \text{ cm}
 \end{aligned}$$

0.6

(Full 0.6 for answer between 23.5 to 24.5) ✓

### C. The Extremum Principle and the Wave Nature of Matter

We now explore if the two approaches, namely, Newton's Second Law of Motion and the PLA, can be connected by invoking the notion of the de Broglie wave associated with a moving particle. For this we assume that a particle moving from O to P can take all possible trajectories and we will seek a trajectory that depends on the constructive interference of de Broglie waves.

- (C1) As the particle moves along its trajectory by an infinitesimal distance  $\Delta s$ , relate the change  $\Delta\phi$  in the phase of its de Broglie wave to the change  $\Delta A$  in the action and the Planck constant.

[0.6]

#### Solution:

From the de Broglie hypothesis

$$\lambda \rightarrow \lambda_{dB} = h/mv$$

0.1

where  $\lambda$  is the de Broglie wavelength and the other symbols have their usual meaning

$$\Delta\phi = \frac{2\pi}{\lambda} \Delta s$$

0.2

$$= \frac{2\pi}{h} mv \Delta s$$

0.1

$$= \frac{2\pi \Delta A}{h}$$

0.2



- (C2) Recall the problem from part A where the particle traverses from O to P (see Fig. 4). Let an opaque partition be placed at the boundary AB between the two regions. There is a small opening CD of width  $d$  in AB such that  $d \ll (x_0 - x_1)$  and  $d \ll x_1$ . Consider two extreme paths OCP and ODP such that OCP lies on the classical trajectory discussed in part A. Obtain the phase difference  $\Delta\phi_{CD}$  between the two paths to first order.

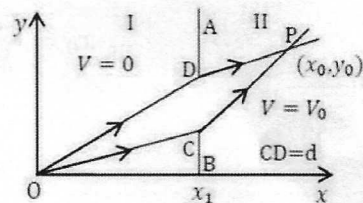
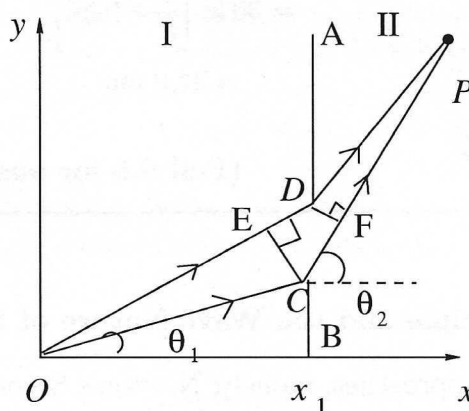


Figure 4

Figure4

**Solution:**



Consider the extreme trajectories  $OCP$  and  $ODP$  of (C1)  
The geometrical path difference is  $ED$  in region I and  $CF$  in region II.  
This implies (note:  $d \ll (x_0 - x_1)$  and  $d \ll x_1$ )

$$\Delta\phi_{CD} = \frac{2\pi d \sin \theta_1}{\lambda_1} - \frac{2\pi d \sin \theta_2}{\lambda_2}$$

0.6

first expression 0.2, second expression 0.2, negative sign 0.2 = 0.6 points.  
Also if negative sign missing then no further credit for the remaining part

$$\begin{aligned} \Delta\phi_{CD} &= \frac{2\pi m v_1 d \sin \theta_1}{h} - \frac{2\pi m v_2 d \sin \theta_2}{h} \\ &= 2\pi \frac{md}{h} (v_1 \sin \theta_1 - v_2 \sin \theta_2) \\ &= 0 \quad (\text{from A2 or B1}) \end{aligned}$$

0.1  
0.1

0.3

0.6

Thus near the classical path there is invariably constructive interference.

exact expression  $\rightarrow$  0.6 ||  $\nexists$  für "meh" als 1. Ordnung!

## Matter Wave Interference

Consider an electron gun at O which directs a collimated beam of electrons to a narrow slit at F in the opaque partition  $A_1B_1$  at  $x = x_1$  such that OFP is a straight line.

6

$P$  is a point on the screen at  $x = x_0$  (see Fig. 5). The speed in I is  $v_1 = 2.0000 \times 10^7 \text{ m s}^{-1}$  and  $\theta = 10.0000^\circ$ . The potential in region II is such that the speed  $v_2 = 1.9900 \times 10^7 \text{ m s}^{-1}$ . The distance  $x_0 - x_1$  is 250.00 mm (1 mm =  $10^{-3} \text{ m}$ ). Ignore electron-electron interaction.

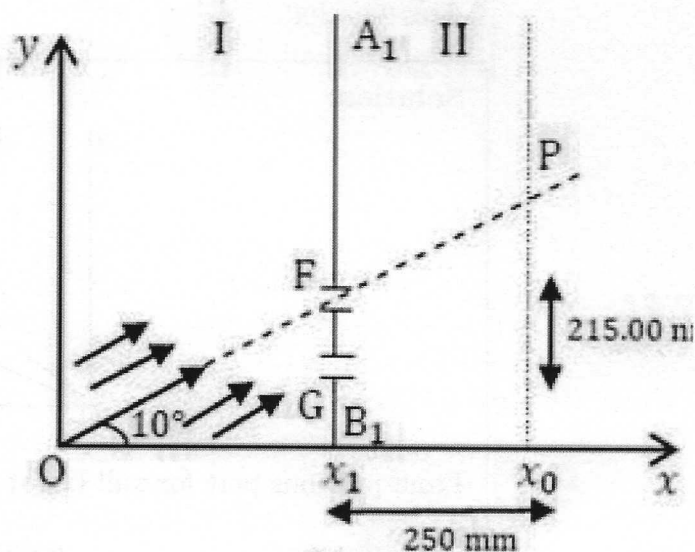


Figure 5

Figure 5

- (D1) If the electrons at O have been accelerated from rest, calculate the accelerating potential  $U_1$ .

[0.3]

**Solution:**

$$\begin{aligned} qU_1 &= \frac{1}{2} mv^2 \\ &= \frac{9.11 \times 10^{-31} \times 4 \times 10^{14}}{2} \text{ J} \\ &= 2 \times 9.11 \times 10^{-17} \text{ J} \\ &= \frac{2 \times 9.11 \times 10^{-17}}{1.6 \times 10^{-19}} \text{ eV} \\ &= 1.139 \times 10^3 \text{ eV} \quad (\simeq 1100 \text{ eV}) \\ U_1 &= 1.139 \times 10^3 \text{ V} \end{aligned}$$

0.2

*full marks*

*1.100 → 1.150*

0.1

- (D2) Another identical slit G is made in the partition  $A_1B_1$  at a distance of 215.00 nm (1 nm =  $10^{-9} \text{ m}$ ) below slit F (Fig. 5). If the phase difference between de Broglie waves arriving at  $P$  from  $F$  and  $G$  is  $2\pi\beta$ , calculate  $\beta$ .

[0.8]

**Solution:** Phase difference at  $P$  is

$$\begin{aligned} \Delta\phi_P &= \frac{2\pi d \sin \theta}{\lambda_1} - \frac{2\pi d \sin \theta}{\lambda_2} \\ &= 2\pi(v_1 - v_2) \frac{md}{h} \sin 10^\circ = 2\pi\beta \\ \beta &= 5.13 \end{aligned}$$

0.2

*full marks*

*5.10 → 5.20*

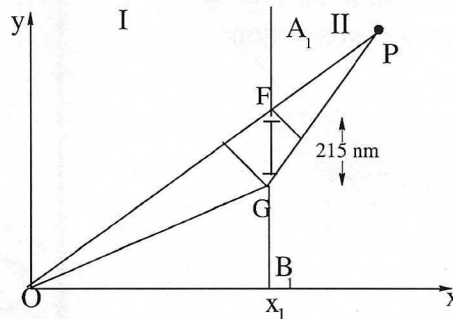
0.4

0.2

- (D3) What is the smallest distance  $\Delta y$  from P at which null (zero) electron detection may be expected on the screen? [Note: you may find the approximation  $\sin(\theta + \Delta\theta) \approx \sin\theta + \Delta\theta \cos\theta$  useful]

[1.2]

**Solution:**



From previous part for null (zero) electron detection  $\Delta\phi = 5.5 \times 2\pi$

$$\therefore mv_1 \frac{d \sin \theta}{h} - \frac{mv_2 d \sin(\theta + \Delta\theta)}{h} = 5.5$$

$$\begin{aligned} \sin(\theta + \Delta\theta) &= \frac{\frac{mv_1 d \sin \theta}{h} - 5.5}{\frac{mv_2 d}{h}} \\ &= \frac{v_1}{v_2} \sin \theta - \frac{h}{m v_2 d} 5.5 \\ &= \frac{2}{1.99} \sin 10^\circ - \frac{5.5}{1374.78 \times 1.99 \times 10^7 \times 2.15 \times 10^{-7}} \\ &= 0.174521 - 0.000935 \end{aligned}$$

This yields  $\Delta\theta = -0.0036^\circ$

The closest distance to P is

$$\begin{aligned} \Delta y &= (x_0 - x_1)(\tan(\theta + \Delta\theta) - \tan \theta) \\ &= 250(\tan 9.9964^\circ - \tan 10^\circ) \\ &= -0.0162 \text{ mm} \\ &= -16.2 \mu\text{m} \end{aligned}$$

(0.2 points deducted if calculation not done in radians)

The negative sign means that the closest minimum is below P.

**Approximate Calculation for  $\theta$  and  $\Delta y$**

Using the approximation  $\sin(\theta + \Delta\theta) \approx \sin\theta + \Delta\theta \cos\theta$

The phase difference of  $5.5 \times 2\pi$  gives

$$mv_1 \frac{d \sin 10^\circ}{h} - mv_2 \frac{d(\sin 10^\circ + \Delta\theta \cos 10^\circ)}{h} = 5.5$$

From solution of the previous part

$$mv_1 \frac{d \sin 10^\circ}{h} - mv_2 \frac{d \sin 10^\circ}{h} = 5.13$$



Therefore

$$mv_2 \frac{d\Delta\theta \cos 10^\circ}{h} = 0.3700$$

This yields  $\Delta\theta \approx 0.0036^\circ$

$\Delta y = -0.0162 \text{ mm} = -16.2 \mu\text{m}$  as before

[0.2]  
[0.2]  
[0.4]

- (D4) The electron beam has a square cross section of  $500 \text{ nm} \times 500 \text{ nm}$  and the setup is  $2 \text{ m}$  long. What should be the minimum beam flux density  $I_{min}$  (number of electrons per unit normal area per unit time) if, on an average, there is at least one electron in the setup at a given time?

[0.4]

**Solution:** The product of the speed of the electrons and number of electron per unit volume on an average yields the intensity.

Thus  $N = 1 = \text{Intensity} \times \text{Area} \times \text{Length} / \text{Electron Speed}$

$$= I_{min} \times 0.25 \times 10^{-12} \times 2/2 \times 10^7$$

This gives  $I_{min} = 4 \times 10^{19} \text{ m}^{-2} \text{ s}^{-1}$

0.2  
0.2

R. Bach, D. Pope, Sy-H Liou and H. Batelaan, New J. of Physics Vol. 15, 033018 (2013).