T2: James Webb Space Telescope (12 pts)

Completed July 13, 4:30 PM China Time

Some general notes for entire Theory 2 marking

An equation which is dimensionally correct, but missing a multiplicative factor or having a single transcription error from a previous equation, will receive a deduction of -0.1 pts.

An equation which is dimensionally incorrect or one which has more than two transcription errors will receive no points.

Follow on errors are not transcription errors; the only penalty will be in the first occurrence of a mistake, except in the case of a dimensionally incorrect equation, which still receives no points, even if a follow on error.

There are two follow on caveats below.

If an error in an equation trivializes the remainder of the problem, then no additional points after that should be awarded. For example, if a student is computing counts, and they arrive at the incorrect answer of zero, then they should not get future points that compute intensity, density, uncertainty, as these would all become trivial.

If an error in an equation makes the remainder of a problem physically unrealistic, then they should get no points for any requested numerical results, but they can continue to get points for theoretical equations. For example, if a student has an extra factor of 100, they can get points for derivations, but if asked to find a temperature they will not get points for reporting 100 times the actual temperature. They will also not get points for reporting the correct actual temperature, because it will not be consistent with their theory.

If an equation can be implied to have been used, then the assumption is that it did exist and would get points. For example, writing Eq. 5 without explicitly writing Eq. 4 would get points for both equations, subject to error rules above.

In places on the mark scheme there are a range of acceptable answers, and in places the range is divided into two possible ranges, a close range for full points, and a larger range for partial points. This might appear like this:

$35\mu\text{m} \le d_d \le 47\mu\text{m}$	0.2 pts
$20\mu\text{m} \le d_d \le 90\mu\text{m}$	0.1/0.2 pts

which means that they get 0.2 pts if they are within the narrow range, but only 0.1 pts if they are outside the narrow range but still within the larger range. They would never get 0.3 pts in this scheme, so don't double count!

Part A: Imaging a Star (1.8 pt)

1. Diameter of image

   The ratio of diameter d_o for an object at a distance $D_o \gg f$ and an image diameter d_i is given by

$$\frac{d_i}{d_o} = \frac{f}{D_o}, \tag{1}$$

so the diameter of the image is

$$\begin{aligned} d_i &= \frac{(1.7\times10^{11}\text{ m})(130\text{m})}{(89\text{ ly})(3\times10^8\text{ m/s})(365\text{ d/y})(86,400\text{ s/d})} = \\ &= 2.6\times10^{-5}\text{ m} = 26\,\mu\text{m}. \end{aligned}$$

Marking scheme:

correct formula Eq 1	0.2 pts
$d_i = (26\pm1)\,\mu\text{m}$	0.2 pts
sum	**0.4pts**

Units must be shown for a numerical result to get points; writing the correct answer without showing work also receives full marks for this problem.

2. Diameter of central maximum

   The angular radius of the central maximum is

$$\theta_{\text{min}} = 1.22\frac{\lambda}{D} \tag{2}$$

$\lambda = 800\text{nm}$ is given in the problem

D is the aperture size, which is the primary mirror, or $\frac{\pi}{4}D^2 = 25\text{ m}^2$, so

$$D = 5.6\text{m}$$

The diameter of the central maximum is then

$$d_d = 2\theta_{\text{min}}f = 2.44\frac{\lambda}{D}f = 1.22\frac{\lambda f}{\sqrt{A/\pi}} \tag{3}$$

The numerical value is

$$\begin{aligned} d_d &= 2(1.22)\frac{(8\times10^{-7}\text{ m})}{(5.6\text{ m})}(130\text{ m}) = \\ &= 4.5\times10^{-5}\text{ m} = 45\,\mu\text{m}. \end{aligned}$$

$d_d = 37\mu\text{m}$ is also acceptable (omitting the factor of 1.22 is okay).

Marking scheme:

correct formula Eq 3	0.1 pts
Aperture $D = (5.6\pm0.2)\text{m}$	0.1 pts
$35\mu\text{m} \le d_d \le 47\mu\text{m}$	0.2 pts
sum	**0.4pts**

No penalty for ignoring factor of 1.22, so check their math. Units must be shown for a numerical result to get points; writing the correct answer without showing work also receives full marks for this problem.

3. Equilibrium temperature of the detector at the location of the image?

   The radiant power from the star is

$$P_g = 4\pi {r_o}^2\sigma {T_g}^4 \tag{4}$$

The intensity at the location of the scope is

$$I_g = \frac{P_g}{4\pi {D_o}^2} = \left(\frac{r_o}{D_o}\right)^2\sigma {T_g}^4 \tag{5}$$

This is collected onto the mirror with area A and focused on a single spot of radius r_i, so that the power incident is

$$P_i = A\left(\frac{r_o}{D_o}\right)^2 \sigma {T_g}^4 = A\left(\frac{r_i}{f}\right)^2 \sigma {T_g}^4 \tag{6}$$

But at the image we have an equilibrium temperature of

$$P_i = a\sigma {T_p}^4,$$

where $a = \pi r_i^2$, so

$$a\sigma {T_p}^4 = \left(\frac{r_i}{f}\right)^2 A\sigma {T_g}^4$$

or, ignoring diffraction,

$$T_p = \left(\frac{A}{\pi f^2}\right)^{\frac{1}{4}} T_g \approx 530\,\mathrm{K} \tag{7}$$

When considering diffraction the actual area of the stars' image is larger,

$$a' = \left(\frac{d_i + d_d}{d_i}\right)^2 a \approx 2.73a \tag{8}$$

where the actual ratio depends on answers above.

This means the actual pixel temperature will be

$$T_{p,\text{correct}} = \left(\frac{A}{(2.73)\pi f^2}\right)^{\frac{1}{4}} T_g \approx 320\mathrm{K}. \tag{9}$$

Marking scheme:

power of source, Eq 4	0.2 pts
intensity at mirror, Eq 5	0.2 pts
power of image, Eq 6	0.2 pts
correct for diffraction Eq. 8	0.1 pts
Either Eq. 7 or Eq. 9	0.1 pts
numerical result	0.2 pts
sum	**1.0 pt**

Units must be shown for a numerical result to get points; the answer $T \approx (320 \pm 10)$K for including diffraction or $T \approx (530 \pm 10)$K for ignoring diffraction must be consistent with their approach. Check the number, since the ratio in Eq 8 depends on their answer to A.2

Students must present a symbolic equation in their solution.

Writing Eq. 9 without showing any other work receives 0.8 pts; Writing Eq. 7 without showing any other work receives 0.7 pts.

Part B: Counting Photons (1.8 pt)

1. Temperature of source

   We are interested in the slope of the graph, which is

$$\text{slope} = -\frac{(3) - (-1)}{(0.11 1/\mathrm{K}) - (0.15 1/\mathrm{K})} = 100\,\mathrm{K}$$

   Since this is a characteristic temperature, it is at least a partial answer to the problem.

   The value of

$$\left|\frac{\Delta E_g}{6k_B}\right| = \ln 10 \times 100\,\mathrm{K} = 230\mathrm{K}$$

   So the value of

$$\Delta E_g = 6 \times 230\mathrm{K} = 1380\mathrm{K}$$

   Marking scheme:

slope of graph = 0.01	0.2 pts
$T_{graph} = 230$K	0.1 pts
$T_{source} = 1380$K	0.1 pts
sum	**0.4 pt**

   Writing either temperature correctly implies they found the slope of graph, and would get the +0.2 pts. Just writing $T_{source} = 1380$K gets full marks, as it really is possible to solve this in one's head.

   Order of magnitude $T = 10^3$K will get full marks, and no work needs to be shown.

2. Write an expression for the total count uncertainty σ_t

   The three uncertainties are

$$\sigma_r$$

   and

$$\sigma_d = \sqrt{i_d \tau}$$

   and

$$\sigma_p = \sqrt{p\tau}$$

   and then

$${\sigma_t}^2 = {\sigma_r}^2 + (i_d + p)\,\tau$$

   Marking scheme:

correct error for dark current	0.1 pts
correct read photon	0.1 pts
added in quadrature	0.2 pts
sum	**0.4 pt**

   Writing

$$\sigma_t = \sigma_r + \sqrt{i_d \tau} + \sqrt{p\tau}$$

   only gets +0.1, instead of the quadrature +0.2

   Correct dark current and photon count errors in final answer are acceptable evidence for those points; it is not necessary for the student to explicitly state what is what.

3. Determine the photon count for a signal to noise ratio of $S/N = 10$.

   At a temperature of $T = 7.5$K, the dark current is $i_d = 5$ electrons/second. This gives a total dark current count of
   $$i_d\tau = 5 \times 10^4$$
   Answers in the range $i_d = 5 \pm 1$ will be accepted for full marks.

   Let P be the photon count. Then
   $$P = 10\sigma_t$$
   so
   $$P^2 = 100\left(\sigma_r^2 + i_d\tau + P\right) \tag{10}$$
   with solution $P \approx 2290$, and a rate of $p = 0.229$ photons per second.

   Marking scheme:

$i_d = (5 \pm 1)$ e/s	0.2 pts
$1 \leq i_d \leq 10$	0.1/0.2 pts
Eq 10	0.1 pts
$0.206 \leq p \leq 0.25$	0.2 pts
$0.10 \leq p \leq 0.33$	0.1/0.2 pts
sum	**0.5 pt**

   They only get the points for p, the count rate, if it agrees with their assumption for i_d, so check the math!

   Writing only the absolute counts P instead of the rate p would get 0.1 pts for $2060 < P < 2500$ and no points if outside this range.

4. What is intensity of source?

   The near-infrared photons have an energy of $E_g = 2.3 \times 6k_BT$, so
   $$E_\lambda = (1380\,\mathrm{K})(1.38 \times 10^{-23}\,\mathrm{J/K}) = 1.9 \times 10^{-20}\,\mathrm{J}$$
   This is not an order of magnitude question like B.1

   The energy received every second is
   $$E = (0.23)(1.9 \times 10^{-20}\,\mathrm{J}) = 4.4 \times 10^{-20}\,\mathrm{J}$$
   and the incident intensity on the primary mirror is then
   $$I = \frac{E/t}{A} = \frac{(4.4 \times 10^{-20}\,\mathrm{J/s})}{(25\,\mathrm{m}^2)} = 1.8 \times 10^{-22}\,\mathrm{W}$$
   Marking scheme:

$E_\lambda = (2 \pm 0.1) \times 10^{-20}$J	0.3 pts
Forgetting ln 10 factor	-0.1 pts
$I = (1.8 \pm 0.2) \times 10^{-22}$ W	0.2 pts
sum	**0.5 pt**

   If they forget factor ln 10, then the correct intensity would be $(7.8 \pm 0.2) \times 10^{-23}$ W. They only get the ln 10 penalty once!

Part C: The Passive Cooling

1. Find expressions for the temperatures of first and fifth sheet

   This is a cleaned up version of an "ideal" solution

   Let Q_i represent heat flow off of a surface, and Q_{ij} represent the heat flow difference off of two surfaces that are facing each other.

   The student needs to consider the three types of differences below:

   Between sun and first sheet:
   $$Q_{01} = \epsilon A\sigma\left(\frac{I_0}{\sigma} - T_1^4\right) \tag{11}$$
   which is the net heat flow into sheet 1 from the sun-side.

   Between any two adjacent sheets:
   $$Q_{ij} = \epsilon A\sigma\left(T_i^4 - T_j^4\right), \tag{12}$$
   which is *not* the net heat flow between the sheets, it is merely a convenient expression to use later.

   Between last sheet and the cold, cruel vacuum of space:
   $$Q_{56} = \epsilon A\sigma\left(T_5^4\right), \tag{13}$$
   which is the net heat flow out of the far side of the last sheet.

   From the problem text, the flux emitted by one sheet and absorbed by an adjacent sheet is
   $$q_i = \alpha Q_i$$
   so that the net heat flow flux out of one sheet absorbed by the adjacent sheet is
   $$q_{ij} = \alpha Q_{ij}$$
   and the flux ejected into space between two sheets is
   $$q'_{ij} = \beta Q_{ij}$$
   This doesn't affect the marking, but the approximation being made here is that β is the same for all four pairs of adjacent sheets. This makes the math solvable, and was the explicit assumption that the students were told to make.

   A student will need to recognize that
   $$\begin{aligned} Q_{01} &= \epsilon A\sigma\left(\frac{I_0}{\sigma} - T_1^4\right) && (14)\\ Q_{12} &= \epsilon A\sigma\left(T_1^4 - T_2^4\right) && (15)\\ Q_{23} &= \epsilon A\sigma\left(T_2^4 - T_3^4\right) && (16)\\ Q_{34} &= \epsilon A\sigma\left(T_3^4 - T_4^4\right) && (17)\\ Q_{45} &= \epsilon A\sigma\left(T_4^4 - T_5^4\right) && (18)\\ Q_{56} &= \epsilon A\sigma\left(T_5^4\right) && (19) \end{aligned}$$
   can be summed to give
   $$Q_{01} + Q_{12} + Q_{23} + Q_{34} + Q_{45} + Q_{56} = \epsilon A I_0 \tag{20}$$

A student will need to consider energy balance across any one sheet:

$$q_{i-1,i} = q_{i,i+1} + q'_{i,i+1} \tag{21}$$

basically stating that the net flow into sheet i from sheet $i-1$ must equal the net flow out of sheet i to either sheet $i+1$ or into space.

Substitute in Q_{ij},

$$\alpha Q_{i-1,i} = \alpha Q_{i,i+1} + \beta Q_{i,i+1}$$

or

$$Q_{i-1,i} = \left(\frac{\alpha+\beta}{\alpha}\right) Q_{i,i+1} \tag{22}$$

The relation for sheet 1 is a little different:

$$q_{0,1} = Q_{0,1} = \alpha Q_{1,2} + \beta Q_{1,2} \tag{23}$$

and so is the relation for sheet 5:

$$q_{4,5} = Q_{5,6} \tag{24}$$

What will eventually matter most is

$$Q_{56} = \frac{\alpha^4}{(\alpha+\beta)^4} \tag{25}$$

Now use the recursion of Eq. 22 to sum up the six Q_{ij} terms in Eq. 20:

$$kQ_{0,1} = \epsilon A I_0, \tag{26}$$

with the constant k defined as

$$k = 1 + \frac{1}{\alpha+\beta} + \frac{\alpha}{(\alpha+\beta)^2} + \frac{\alpha^2}{(\alpha+\beta)^3} + \frac{\alpha^3}{(\alpha+\beta)^4} + \frac{\alpha^4}{(\alpha+\beta)^4} \tag{27}$$

Substitute the expression for $Q_{0,1}$ back into Eq. 11 and get

$$T_1 = \sqrt[4]{\frac{I_0}{\sigma}\left(1 - \frac{1}{k}\right)} = \sqrt[4]{\frac{I_0}{k\sigma}(k-1)} \tag{28}$$

and the into Eq. 25 and Eq. 13 to get

$$T_5 = \frac{\alpha}{\alpha+\beta}\sqrt[4]{\frac{I_0}{k\sigma}} \tag{29}$$

which can also be written elegantly as

$$T_5 = \frac{\alpha}{\alpha+\beta}\sqrt[4]{\frac{1}{k-1}}\, T_1.$$

Marking scheme:

Net flow into sheet 1 Eq 11	0.2 pts
"Net" flow sheet $i \to j$ Eq 12	0.2 pts
Net flow out of sheet 5 Eq 13	0.2 pts
Sum to eliminate sheet temps Eq 20	0.2 pts
Generic Energy flow Eq 21	0.2 pts
Recursion for Q_{ij} Eq 22	0.2 pts
Sheet 1 Energy flow Eq 23	0.2 pts
Sheet 5 Energy flow Eq 24	0.2 pts
Simplify sum Eq 26	0.2 pts
Find k Eq 27	0.2 pts
Final Expression for T_1, Eq 28	0.2 pts
Final Expression for T_5, Eq 29	0.2 pts
sum	**2.4 pt**

- In most cases a single mistake in an equation that is still dimensional correct will get 0.1 pts for the equation. Making the same mistake multiple times is not a follow on error, and would be penalized every time.
- Any equivalent to Eq 12 would get the 0.2 pts.
- Any attempt to balance energy flow on a generic sheet like Eq 21 that is dimensionally correct and reasonable given their presentation would get the 0.2 pts
- Since sheet 1 and sheet 5 have a different energy balance approach, they must show those separately to get those points.
- It is possible to arrive at Eq 26 based on dimensional analysis alone. A student who writes some form of Eq 26 without clear justification would get points for Eq 20 and Eq 26. The could get full marks for final sheet temperatures if it is consistent; if they did, then they would probably also get at least partial points for Eq 11 and/or Eq 13. They would need to introduce one more unknown constant to have defined $Q_{56} = k' Q_{01}$. The maximum points I would expect with this approach is 1.2 pts.
- k in Eq 27 is allowed a single error for 0.1 pts. Two errors is no points.
- Failing to include the back flux of Eq 12 is only a penalty on that equation but would be zero points, as it is a serious error. That means writing the equivalent of $Q_{ij} = \epsilon A T_i^4$ is zero points! The work after this would have a follow on error that would need to be traced.

Original Solution

Don't use this, eh?

Start with a statement of net energy flow q_{01} into the first sheet from the sun:

$$q_{01} = \epsilon A \left(I_0 - \sigma T_1^4\right) \tag{30}$$

where A is the area of the sheet, ϵ is the emissivity, σ is the Stefan-Boltzman constant, and T_1 is the temperature of the first sheet.

Now consider the space between two sheets i and j. Each sheet radiates an energy flow

$$\epsilon A \sigma T^4$$

toward the other sheet, but a fraction β is ejected into space out the gap.

We have defined α as the fraction emitted from one sheet that is absorbed by the other sheet, so the net energy flow from sheet i into sheet j is

$$q_{ij} = \alpha \epsilon A \sigma \left({T_i}^4 - {T_j}^4\right) \tag{31}$$

There is also a lost fraction emitted into space from between the sheets, given by

$$q'_{ij} = \beta \epsilon A \sigma \left({T_i}^4 - {T_j}^4\right) = \frac{\beta}{\alpha} q_{ij} \tag{32}$$

Don't make the mistake of assuming that $\alpha + \beta = 1$, as some of the energy emitted from a sheet could be reabsorbed by that sheet.

Finally, write an expression for the net thermal radiant energy flow into space, with an ambient temperature of $T_{space} = 0$, from the far side of the fifth sheet.

$$q_{5s} = \epsilon A \left(\sigma T_5^4 - \sigma T_s^4\right) = A \epsilon \sigma T_5^4 \tag{33}$$

Write each of the Eq. 31, above in the form

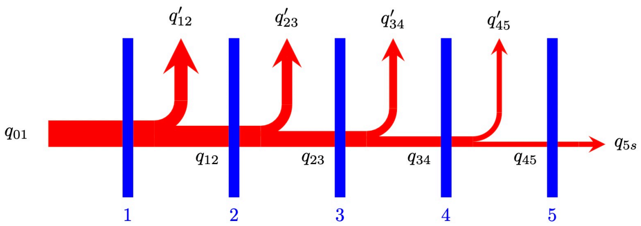
$$\frac{1}{\alpha} q_{ij} = A \epsilon \sigma (T_i^4 - T_j^4), \tag{34}$$

and then sum up the terms from Eq. 30, the four from Eqs. 34, and Eq. 33:

$$q_{01} + \frac{1}{\alpha}\left(q_{12} + q_{23} + q_{34} + q_{45}\right) + q_{5s} = \epsilon A I_0 \tag{35}$$

as all of the T_i terms cancel out on the right!

Now consider a schematic of the energy flow below



From energy conservation, the net flow into sheet one from the sun and the net flow out of sheet one toward sheet two or ejected from gap is

$$q_{01} = q_{12} + q'_{12}, \tag{36}$$

where q'_{12} is the part emitted into space from the gap.

Combine with Eq. 32 and

$$q_{01} = \left(1 + \frac{\beta}{\alpha}\right) q_{12} = \frac{\alpha+\beta}{\alpha} q_{12} \tag{37}$$

Similarly, for the remaining pairs of sheets,

$$q_{23} = \frac{\alpha}{\alpha+\beta} q_{12} = \left(\frac{\alpha}{\alpha+\beta}\right)^2 q_{01},$$

and

$$q_{34} = \frac{\alpha}{\alpha+\beta} q_{23} = \left(\frac{\alpha}{\alpha+\beta}\right)^3 q_{01},$$

and

$$q_{45} = \frac{\alpha}{\alpha+\beta} q_{34} = \left(\frac{\alpha}{\alpha+\beta}\right)^4 q_{01},$$

Finally, for the fifth (last) sheet all of the net energy flow in from the fourth sheet must be completely ejected into space on the dark side.

$$q_{5s} = q_{45} = \left(\frac{\alpha}{\alpha+\beta}\right)^4 q_{01}. \tag{38}$$

The sum on the left side of Eq. 35 can then be written as

$$k q_{01} = \epsilon A I_0 \tag{39}$$

where

$$k = 1 + \frac{1}{\alpha+\beta} + \frac{\alpha}{(\alpha+\beta)^2} + \frac{\alpha^2}{(\alpha+\beta)^3} + \frac{\alpha^3}{(\alpha+\beta)^4} + \frac{\alpha^4}{(\alpha+\beta)^4}$$

is a convenient constant.

Combining Eq. 30 with Eq. 39,

$$\frac{\epsilon A I_0}{k} = \epsilon A \left(I_0 - \sigma T_1^4\right)$$

so

$$T_1 = \sqrt[4]{\frac{I_0}{\sigma}\left(1 - \frac{1}{k}\right)} = \sqrt[4]{\frac{I_0}{k\sigma}(k-1)} \tag{40}$$

From above,

$$q_{5s} = \left(\frac{\alpha}{\alpha+\beta}\right)^4 q_{01}.$$

so

$$A\epsilon\sigma T_5^4 = \left(\frac{\alpha}{\alpha+\beta}\right)^4 \frac{\epsilon A I_0}{k}$$

or

$$T_5 = \frac{\alpha}{\alpha+\beta}\sqrt[4]{\frac{I_0}{k\sigma}} \tag{41}$$

which can also be written elegantly as

$$T_5 = \frac{\alpha}{\alpha+\beta}\sqrt[4]{\frac{1}{k-1}}\, T_1.$$

As this part of the question is complex, with multiple ways to go wrong, and many opportunities for approximations, the marking scheme will be necessarily convoluted.

Some expected mistakes:

(a) Failing to account for the back flux of energy. This would be

$$I_0 = 2\sigma T_1^4$$

and then

$$\alpha\sigma T_1^4 = 2\sigma T_2^4,$$

and so on, concluding with

$$I_0 = \sigma\left(\frac{2}{\alpha}\right)^4 T_5$$

or

$$T_5 = \frac{\alpha}{2} T_1$$

(b) Inconsistent treatment of emissivity
The most likely error is of the form

$$\epsilon I_0 = \sigma T_1^4$$

(c) Incorrectly resolving β and α.

2. Find α and β

Assuming students grab the hint about effective absorptive areas, then expect

Area of gap:

$$A_{\text{gap}} = 4h\sqrt{A_{\text{sheet}}} \tag{42}$$

Area of one sheet A

Assume that the probability of being absorbed by a sheet is the ratio of effective areas

$$\alpha = \frac{\epsilon A_{\text{sheet}}}{2\epsilon A_{\text{sheet}} + A_{\text{gap}}} \tag{43}$$

This result yields $\alpha = 0.3$.

Assume the probability of ejection is a ratio of effective areas

$$\alpha = \frac{A_{\text{gap}}}{2\epsilon A_{\text{sheet}} + A_{\text{gap}}} \tag{44}$$

This result yields $\beta = 0.4$.

Marking Scheme:

Gap area Eq 42	0.2 pts
Estimating α Eq 43	0.2 pts
Estimating β Eq 44	0.2 pts
Factor of 2 for A in both	0.2 pts
Weighting A by emissivity in both	0.2 pts
Finding α	0.1 pts
$0.25 \le \alpha \le 0.35$	0.1 pts
Finding β	0.1 pts
$0.3 \le \beta \le 0.83$	0.1 pts
subtotal	**1.4 pt**
Find a better β	0.2 pts
sum	**1.6 pt**

- "in both" means that to get the points they must have used the factor of two and the emissivity both times; if it is missing from one, they get 0.1 pts for the equation it is present in.
- Find α and β means that it is consistent with own work.
- Assuming $2\alpha + \beta \approx 1$ with proof would mean they only need to find either α or β, and they would get all of the points upon finding the other one. The highest possible subtotal score in the case would be 1.4 pts. Proof can be simple, however, like saying "two sheets, equal probability of being transmitted or absorbed into the other."

- Assuming $2\alpha + \beta \approx 1$ without proof would mean they only need to find either α or β, and they would get 2/3 of the points upon finding the other one. The highest possible subtotal score in the case would be 1.2 pts.
- Assuming $\alpha + \beta \approx 1$ stating a "reasonable" proof would mean they only need to find either α or β, and they would get 2/3 of the points upon finding the other one. The highest possible subtotal score in the case would be 1.2 pts. Stating "energy conservation" is reasonable, though incomplete.
- Assuming $\alpha + \beta \approx 1$ without any proof would mean they only need to find either α or β, and they would get 1/2 of the points upon finding the other one. The highest possible subtotal score in the case would be 1.1 pts.
- Finding α and β means that it is consistent with own work.
- Read the special note about finding a better β below to understand the last 0.2 pts.

Special note

Our assumption is that the rejected heat can be written as

$$q_{ij} = \beta\epsilon\sigma A \left(T_i^4 - T_j^4\right)$$

This is certainly true, but β would be infinite in the case of $T_i = T_j$. It would have been better to write

$$q_{ij} = \beta'\epsilon\sigma A \left(T_i^4 + T_j^4\right)$$

which would follow the energy conservation rule $2\alpha + \beta' = 1$ if $\epsilon \ll 1$.

In fact, β' is really what the student is finding in the approach above.

Assuming that the temperatures of adjacent sheets are related by

$$T_j = \gamma T_i$$

then

$$\beta = \beta' \frac{1+\gamma^4}{1-\gamma^4}$$

In our case,

$$\gamma^4 \approx \frac{(100\text{K})}{(400\text{K})}$$

which means

$$\beta = \frac{5}{3}\beta' = \frac{5}{3} - \frac{10}{3}\alpha$$

is the best estimate; in our case, we expect $\beta = 0.67$.

Any who correctly does this gets those 0.2 pts. If they make a single mistake, but still end up with

$$1 - \alpha > \beta > 1 - 2\alpha$$

they can still get 0.2 pts. If they make two or three mistakes, but still end up with

$$1 - \alpha > \beta > 1 - 2\alpha$$

they can still get 0.1 pts. They only get these points for an effort to deal with our odd definition, and recognizing that the back flux is positive for ejection from the gap. Just writing a different β without justification doesn't get these "special" points.

In the event that a student derives

$$\beta = \frac{5}{3}\beta' = \frac{5}{3} - \frac{10}{3}\alpha$$

Then their minimum score for C.3 should be 0.8 pts, then subtract off 0.1 pt for every error in their derivation if their answer is close. After that, look back at their work on estimating α or β alone, and add on half points for any success, up to 1.4 pts (or 1.6 pts, if no mistakes). The score they get for C.3 would be the larger of the two scoring approaches.

Original Solution

These might still apply in some cases; the first path was rewritten above, so not included, and the second path assumed reflective sheets at angles, so was deleted. Choice C is a variation that can yield a correct value for β, but it would need to be combined with some other approach to find α.

Choice C: Estimate the radiant flux from the gap

Assuming that the enclosed volume is a black body in equilibrium, which it isn't, at a temperature equal to a quartic averaging of the two temperatures: $\frac{1}{2}(T_i^4 + T_j^4)$. Then the energy is radiated out of the area according to

$$q_{lost} = \sigma A_g \frac{1}{2}(T_i^4 + T_j^4)$$

where A_g is the area of the gap, given by

$$A_g = 4h\sqrt{A}$$

But energy was entering the region at the rate

$$q_{in} = \epsilon\sigma A(T_i^4 + T_j^4),$$

so the fraction lost is

$$\beta = \frac{A_g}{2A} = \frac{2h}{\epsilon\sqrt{A}} = 0.7$$

Marking Scheme:

Estimating flux out of gap	0.2 pts
Exact flux into volume	0.2 pts
Correct estimate of gap area	0.2 pts
Finding β	0.1 pts
$0.65 \le \beta \le 0.75$	0.1 pts
sum	**0.8 pt**

The bounds on allowed values for β are smaller in this approach, because there really is only one reasonable answer.

Look back at the full solution to see how to score estimates for α based on this β.

Note that this approach has fewer possible points, as the expression for the flux out of gap makes an assumption that is based on unchecked physics.

Choice D: Another Approach?

Surely there will be some creative students who show other approaches. We will try and expand the marking scheme to recognise these approaches as soon as they occur. A rough guide for an incomplete approach is

Tentative Marking Scheme:

Relevant correct physics equation, each	0.2 pts
Reasonable approximation, each	0.1 pts

The maximum possible is still 1.6 pts.

An equation is only relevant if it can be argued that it would lead to an answer to the question within the bounds of the approach that they are following. For example, don't award points for both counting bounces and effective surfaces, unless each equation contributes to a unified approach that would lead to the answer. Find the most rewarding approach, and award points for that line of reasoning.

If a student only finds one of α or β, then they get 0.2 pts for the first. The marking scheme assumed they would look for α first, but they might have looked for β, and only found that.

Be very careful with mixing and matching approaches!

A student will not get half the points for one approach plus half the points for another approach if they attempt, but don't succeed, with both approaches. They will be awarded the higher of the two scores, not the sum.

3. Numerically determine the temperature of sheet 1 and the temperature of sheet 5.

The solar intensity is $I_0 = 1360\,\mathrm{W/m^2}$, the background temperature of space is $T_b = 20\,\mathrm{K}$ and is negligible.

Assuming a student does C.1 correctly, and uses $2\alpha + \beta = 1$, then

β	α	T_1 (K)	T_2 (K)
0.3	0.35	383	120
0.4	0.3	380	102
0.5	0.25	376	83
0.6	0.2	373	65
0.7	0.15	369	48

The other bound is $\alpha + \beta = 1$, in that case:

β	α	T_1 (K)	T_2 (K)
0.3	0.7	370	189
0.4	0.6	368	165
0.5	0.5	365	140
0.6	0.4	363	114
0.7	0.3	361	87

The numbers agree well with the theoretical performance of 320 K and 90 K. Some of the major differences are explained by different coatings on different surfaces, a temperature and wavelength dependence on emissivity that is designed to reflect visible light from the sun while radiating infrared on the sunside of sheet 1, and the sheets are not uniform temperature.

Marking Scheme:

T_1 consistent with own formula	0.1 pts
$250\mathrm{K} \le T_1 \le 400\mathrm{K}$	0.1 pts
T_5 consistent with own formula	0.1 pts
$45\mathrm{K} \le T_5 \le 200\mathrm{K}$	0.1 pts
sum	**0.4 pt**

The grade depends on self consistency with the previous work, so the numbers must be checked!

Note that here is a case where follow on errors could be penalized twice; students should recognize that an answer is not reasonable, as T_1 should be on the order of the temperature of the Earth, and that T_5 ought to have shown significant, but not incredible, cooling.

Part D: The Cryo-Cooler

1. What state variables change?

(a) In order to force the gas through the plug, which offers up considerable viscous friction, $P_1 > P_2$; it is this pressure difference that is the source of the force.

(b) Viscous friction is dissipative, so the internal energy of the gas must decrease as it moves through the plug, and then $U_1 > U_2$.

(c) Though no heat is gained or lost, this is not a constant entropy process; that can be seen because it is an irreversible process. As such, $S_1 < S_2$

(d) Since the process of moving across a pressure gradient imparts kinetic energy to an object, it is expected that the fluid velocity on the right will be higher than the left. Since mass is conserved, the volume of a mole of gas on the right must also be higher than the volume of a mole on the left, and $V_1 < V_2$.

(e) The correct answer is $T_1 ? T_2$. If this were an ideal gas, $T_1 > T_2$ since $U \propto T$. But this is not an ideal gas, and U will be a function of temperature and density. As such, it is not possible to know the comparative relation between T_1 and T_2. That's the whole point of this problem, and the challenge of trying to make liquid helium.

Marking scheme:

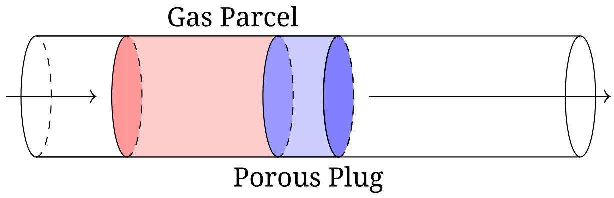
For each correct response	+0.2 pts
sum	**1.0 pt**

Explanations by the students are not needed.

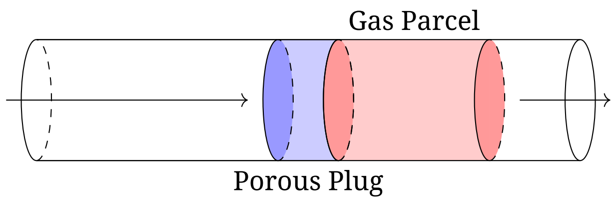
2. A mole of gas at P_1, V_1, T_1, U_1 enters the porous plug from the left, and that mole of gas exits the porous plug on the other side at P_2, V_2, T_2, U_2.

Consider first a control volume approach

The figure below shows the motion of a mole of gas through the plug; the mole is shown in pink. Gas to the left of the mole pushes the mole through the plug with a constant force $P_1 A$ through a volume V_1.



The mole of gas moves through the plug to the right hand side, in the process pushing on the air to the right of the mole with a constant force $P_2 A$, through a volume V_2.



The work that the surrounding gas in region 1 does on the gas pushing it into the plug is

$$W_1 = P_1 V_1$$

because the pressure is constant, and the effective change of volume is V_1. Similarly, when the gas enters region 2 it must displace a volume V_2 of gas that was already there, so

$$W_2 = -P_2 V_2$$

The net work is then

$$W_{net} = P_1 V_1 - P_2 V_2 \tag{45}$$

Since there is no heat exchanged,

$$U_2 - U_1 = \Delta U = Q + W_{net} = P_1 V_1 - P_2 V_2 \tag{46}$$

which implies

$$\Delta U = U_2 - U_1 = P_1 V_1 - P_2 V_2.$$

Upon rearranging

$$U_2 + P_2 V_2 = U_1 + P_1 V_1$$

and therefore

$$U + PV$$

is a conserved quantity.

Marking scheme:

Compute correct W_1	0.1 pts
Compute correct W_2	0.1 pts
Write energy law, Eq 46	0.2 pts
Show $U + PV$ conserved	0.2 pts
sum	**0.6 pt**

Consider instead a differential approach

Another way to look at this problem is to focus on a differential sample of gas as it moves through the plug.

The figure below illustrates this

The total energy of parcel of molar size δm has two relevant energy terms: the internal energy δU and the bulk kinetic energy δK. It has a volume δV. These four quantities are extrinsic, but to simplify notation, we will drop the δ. It's still there, just invisible.

For simplicity's sake, assume a cylindrical shape to the parcel, with an end cap area δA and a length dx. Once again, we will drop the δ. There are three forces that act on the shape, one associated with pressure on the left end, one associated with pressure on the right end, and frictional force associated with viscosity against the walls of the container.

Since this is a parcel of differential length dx, the net force associated with the pressure difference between the ends is

$$F_{ends} = -V \frac{dP}{dx}$$

where V is again the volume of the cylinder.

But this force is (mostly) balanced by the viscous frictional force F_{walls} with the walls of the sponge; these two forces effectively add to zero. In fact, it is the viscous forces with the wall that cause the pressure gradient across the sponge.

The bulk kinetic energy of the parcel does not change significantly as it moves through the sponge. This is seen in that the bulk speed of the gas doesn't change significantly as it moves through the sponge.

The problem with this approach is that the system is not in thermodynamic equilibrium; the process is not reversible, so it is not possible to attach well defined state variables. This means that

$$dU = TdS - PdV \tag{47}$$

is not a function that can be integrated; in fact, $dS \neq 0$ from the previous part of the problem. Arguing that $VdP = -TdS$ is rather handwavy, and resolving this actually requires considering a control volume approach.

Still, the energy conservation ideas still hold true, even if thermodynamically poorly defined, so

$$dU = -PdV - VdP$$

since the part associated with $-VdP$ doesn't change the bulk kinetic energy, and instead dissipates into internal energy of the gas.

The result is that

$$dU = -d(PV)$$

or

$$U + PV$$

is a constant

Marking scheme:

Traditional $\delta W = -PdV$	0.1 pts
Bulk kinetic $\delta K = -VdP$	0.1 pts
Explain where δK goes	0.1 pts
Differential Eq 47	0.1 pts
integrate $U + PV$ constant	0.1 pts
sum	**0.5/0.6 pt**

Because of the many subtle traps, this approach will not get the same number of points as the control volume approach.

3. One can find pressure on this graph by applying

$$dU = TdS - PdV$$

and then requiring constant entropy so that $dS = 0$, and then

$$P = -\left(\frac{\partial U}{\partial V}\right)_S \tag{48}$$

which are the negative slopes of the constant entropy curves on a $U - V$ graph.

Then

$$U + PV = U - \left(\frac{\partial U}{\partial V}\right)_S V$$

is the conserved quantity.

Now $-(\partial U/\partial V)_S$ is measured only at the point V_1, U_1, and is the slope of the tangent line to the constant entropy curve. Following that tangent line back a distance V takes it to an intercept with the U axis, and that intercept is then the conserved quantity.

More mathematically, define a function H

$$H = U + PV$$

then

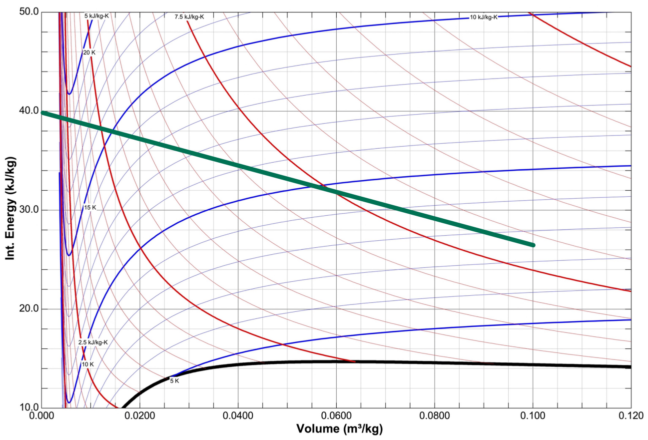
$$U = H_2 - P_2 V$$

is the equation of a line,

$$U = H_2 + \left(\frac{\partial U}{\partial V}\right)_S \bigg|_2 V \tag{49}$$

with the U intercept equal to the conserved H_2.

An estimate can be made visually, but it is difficult to be accurate. Try constructing a line from the point $V_2 = 0.120$, $T_2 = 7.5$ that is tangent to the local isentrope, and the result will intercept the U axis. This result is somewhere around 40. This is shown in green below.

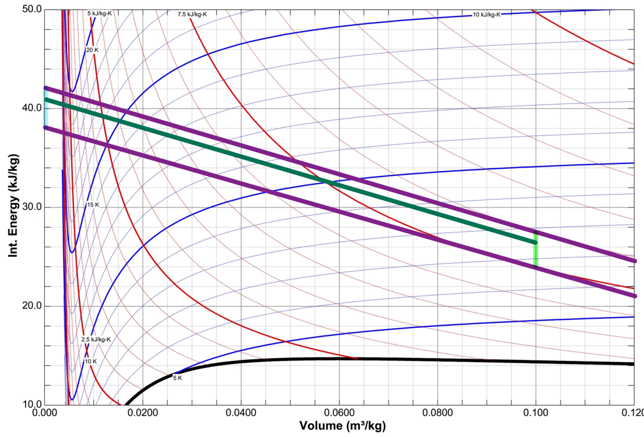


Now to improve the result.

Draw a line out from 39 that is tangent to the nearest isentrope to $V_2 = 0.100$, $T_2 = 7.5$; draw another line out from 41 that is also tangent to the nearest isentrope to $V_2 = 0.100$, $T_2 = 7.5$. These are shown in purple below.

Measuring the distance with a ruler, find the fractional distance between the two purple lines to the point $V_2 = 0.120$, $T_2 = 7.5$ along the highlighted green line. It is about 75% the way from the bottom purple line. This means that the conserved quantity ought be 75% the way up on the highlighted blue section on the graph. A line connecting the two is shown in green.

This point is about 41 kJ/kg. The actual value for the conserved quantity is $U + PV = 40.7$ kJ/kg.



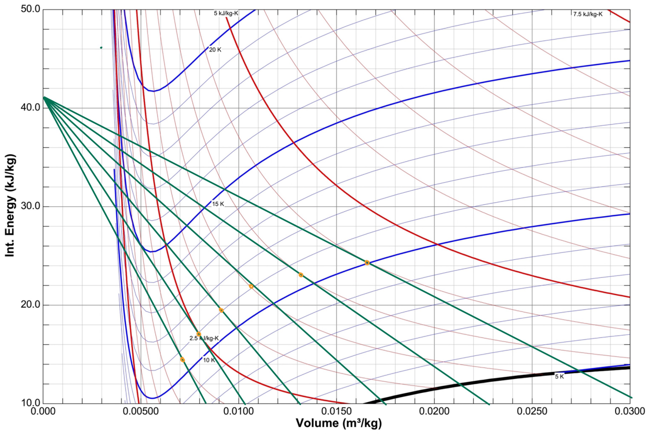
Marking scheme:

Pressure formula stated, Eq 48	0.2 pts
Tangent intercept concept	0.4 pts
A first estimate for H	0.2 pts
Upper bound for estimate set	0.2 pts
Upper bound for estimate set	0.2 pts
Interpolated estimate set	0.2 pts
$40.5 < H < 41.0$	0.2/0.2 pts
$40.2 < H < 41.2$	0.1/0.2 pts
sum	**1.4 pt**

As the task asks for a graphical construction, and it is not possible to construct an accurate tangent to the isentrope at $T_2 = 7.5$K based on a single line, students *must* do something to improve or verify the result, even if it is correct on the first guess. Hence the upper and lower bound approach and interpolation, or something equivalent.

4. Draw a series radial lines out from the conserved point that are tangent to lines of constant entropy. Mark the tangent point. Connect with a smooth curve; this curve is the set of points U_1 as a function of V_1 that has the conserved quantity. Look for the maximum temperature intercept.

   This happens at about $T_1 = 11$K. If T_1 is higher than this, it would not be possible to cool down to $T_2 = 7.5$K.



Students don't need to draw every line, as with a straight edge one can find the tangent that maximizes the temperature T_1 by shifting it around visually.

Line starts from student's H	0.2 pts
Line intercepts an isentrope	0.2 pts
The isentrope matches max T_1	0.2 pts
Stated T_1 within 0.5K of student's construction	0.1 pts
$10\text{K} \leq T_1 \leq 12\text{K}$	0.1 pts
sum	**0.8 pt**

5. Using the slope of the line from the conserved quantity to the maximum temperature point, compute the pressure.

   Using the results from above,

$$P_1 = -\frac{(41) - (10)}{(0) - (0.0170)} = 1.8\,\text{MPa}$$

   If they didn't know to use slope by this point, they can't generate an answer. As such, they would already have received points for the pressure formula, and we only consider the numerical result

P agrees with the slope of the graph	0.1 pts
$1.6\,\text{MPa} \leq P_1 \leq 2.4\,\text{MPa}$	0.1 pts
sum	**0.2 pt**