Solutions

A single spherical silver nanoparticle

2.1	Volume of the nanoparticle: $V=\frac{4}{3}\pi R^3=4.19\times10^{-24}\ \mathrm{m}^3$. Mass of the nanoparticle: $M=V\rho_{\mathrm{Ag}}=4.39\times10^{-20}\ \mathrm{kg}$. Number of ions in the nanoparticle: $N=N_A\frac{M}{M_{\mathrm{Ag}}}=2.45\times10^5$. Charge density $\rho=\frac{eN}{V}=9.38\times10^9\ \mathrm{C\ m^{-3}}$, charge density $\rho=en$. Electrons' concentration $n=\frac{N}{V}=5.85\times10^{28}\ \mathrm{m}^{-3}$. Total charge of free electrons $Q=eN=3.93\times10^{-14}$ C. Total mass of free electrons $m_0=m_eN=2.23\times10^{-25}$ kg.	0.7

The electric field in a charge-neutral region inside a charged sphere

2.2	For a sphere with radius R and constant charge density ρ, for any point inside the sphere designated by radius-vector $\mathbf{r}=r\mathbf{e}_r$ $(r<R)$ Gauss's law yields directly $4\pi r^2\varepsilon_0\boldsymbol{E}_+=\frac{4}{3}\pi r^3\rho\,\boldsymbol{e}_r$, where $\boldsymbol{e}_r$ is the unit radial vector pointing away from the center of the sphere. Thus, $\boldsymbol{E}_+=\frac{\rho}{3\varepsilon_0}\boldsymbol{r}$. Likewise, inside another sphere of radius R_1 and charge density $-\rho$ the field is $\boldsymbol{E}_-=\frac{-\rho}{3\varepsilon_0}\boldsymbol{r}'$, where $\boldsymbol{r}'$ is the radius-vector of the point in the coordinate system with the origin in the center of this sphere. Superposition of the two charge configurations gives the setup we want with $\boldsymbol{r}'=\boldsymbol{r}-\boldsymbol{x}_d$. So inside the charge-free region $\lvert\boldsymbol{r}-\boldsymbol{x}_{\mathrm{p}}\rvert<R_1$ the field is $\boldsymbol{E}=\boldsymbol{E}_++\boldsymbol{E}_-=\frac{\rho}{3\varepsilon_0}\mathbf{r}+\frac{-\rho}{3\varepsilon_0}(\boldsymbol{r}-\boldsymbol{x}_d)$ or $\boldsymbol{E}=\frac{\rho}{3\varepsilon_0}\boldsymbol{x}_d$ with pre-factor $A=\frac{1}{3}$	1.2

The restoring force on the displaced electron cloud

2.3	With $\boldsymbol{x}_{\mathrm{p}}=x_{\mathrm{p}}\,\boldsymbol{e}_x$ and $x_{\mathrm{p}}\ll R$ we have from above that approximately the field induced inside the particle is $\boldsymbol{E}_{\mathrm{ind}}=\frac{\rho}{3\varepsilon_0}\boldsymbol{x}_{\mathrm{p}}$. The number of electrons on the particle's border that produced $\boldsymbol{E}_{\mathrm{ind}}$ is negligibly smaller than the number of electrons inside the particle, so $\boldsymbol{F}\cong Q\boldsymbol{E}_{\mathrm{ind}}=(-eN)\frac{\rho}{3\varepsilon_0}\boldsymbol{x}_{\mathrm{p}}=-\frac{4\pi}{9\varepsilon_0}R^3e^2n^2\,x_p\boldsymbol{e}_x$ (note the antiparallel attractive force is proportional to the displacement that it is similar to Hooke's law). The work done on the electron cloud to shift it is $W_{\mathrm{el}}=-\int_0^{x_{\mathrm{p}}}F(x')\,\mathrm{d}x'=\frac{1}{2}\left(\frac{4\pi}{9\varepsilon_0}R^3e^2n^2\right)x_{\mathrm{p}}^2$	1.0

The spherical silver nanoparticle in an external constant electric field

2.4	Inside the metallic particle in the steady state the electric field must be equal to 0. The induced field (from 2.2 or 2.3) compensates the external field: $\boldsymbol{E}_0+\boldsymbol{E}_{\mathrm{ind}}=0$, so	0.6

	$x_p = \frac{3\varepsilon_0}{\rho} E_0 = \frac{3\varepsilon_0}{en} E_0$. Charge displaced through the yz-plane is the total charge of electrons in the cylinder of radius R and height x_p: $-\Delta Q = -\rho\, \pi R^2 x_p = -\pi R^2\, ne\, x_p$.	

The equivalent capacitance and inductance of the silver nanoparticle

2.5a	The electric energy W_{el} of a capacitor with capacitance C holding charges $\pm\Delta Q$ is $W_{el} = \frac{\Delta Q^2}{2C}$. The energy of such capacitor is equal to the work (see 2.3) done to separate the charges (see 2.4), thus $C = \frac{\Delta Q^2}{2W_{el}} = \frac{9}{4}\varepsilon_0 \pi R = 6.26 \times 10^{-19}$ F.	0.7
2.5b	Equivalent scheme for a capacitor reads: $\Delta Q = CV_0$. Combining charge from (2.4) and capacitance from (2.5a) gives $V_0 = \frac{\Delta Q}{C} = \frac{4}{3} R\, E_0$.	0.4

2.6a	The kinetic energy of the electron cloud is defined as the kinetic energy of one electron multiplied by the number of electrons in the cloud $W_{kin} = \frac{1}{2} m_e v^2 N = \frac{1}{2} m_e v^2 \left(\frac{4}{3}\pi R^3\, n\right)$. The current I is the charge of electrons in the cylinder of area πR^2 and height $v\Delta t$ divided by time Δt (or simply the time derivative of charge $-\Delta Q$), thus $I = -e\, nv\, \pi R^2$.	0.7
2.6b	The energy carried by current I in the equivalent circuit with inductance L is $W = \frac{1}{2} L\, I^2$ is, in fact, the kinetic energy of electrons W_{kin}. Taking the energy and current from (2.6a) gives $L = \frac{4\, m_e}{3\pi R n e^2} = 2.57 \times 10^{-14}$ H.	0.5

The plasmon resonance of the silver nanoparticle

2.7a	From the LC-circuit analogy we can directly derive $\omega_p = (LC)^{-1/2} = \sqrt{ne^2/3\varepsilon_0 m_e}$. Alternatively it is possible to use the harmonic law of motion in (2.3) and get the same result for the frequency.	0.5
2.7b	$\omega_p = 7.88 \times 10^{15}$ rad/s, for light with angular frequency $\omega = \omega_p$ the wavelength is $\lambda_p = 2\pi c/\omega_p = 239$ nm.	0.4

The silver nanoparticle illuminated with light at the plasmon frequency

2.8a	The velocity of an electron $v = \frac{dx}{dt} = -\omega x_0 \sin \omega t = v_0 \sin \omega t$. The time-averaged kinetic energy on the electron $\langle W_k \rangle = \langle \frac{m_e v^2}{2} \rangle = \frac{m_e}{2} \langle v^2 \rangle$. During time τ each electron hits an ion one time. So the energy lost in the whole nanoparticle during time τ is $W_{heat} = N \langle \frac{m_e v^2}{2} \rangle = \frac{4}{3}\pi R^3 n \langle \frac{m_e v^2}{2} \rangle$. Time-averaged Joule heating power $P_{heat} = \frac{1}{\tau} W_{kin} = \frac{1}{2\tau} m_e \langle v^2 \rangle \left(\frac{4}{3}\pi R^3\, n\right)$. The expression for current is taken from (2.6a), squared and averaged	1.0

	$\langle I^2\rangle = (en\,\pi R^2)^2\,\langle v^2\rangle = \left(\frac{3Q}{4R}\right)^2\langle v^2\rangle$.	
2.8b	The average time between the collisions is $\tau \gg 1/\omega_{\mathrm{p}}$, so each electron oscillates many times before it collides with an ion. The oscillating current $I = I_0\sin\omega t = \pi R^2 nev_0\sin\omega t$ produces the heat in the resistance R_{heat} equal to $P_{\mathrm{heat}} = R_{heat}\langle I^2\rangle$, that together with results from (2.8a) leads to $R_{\mathrm{heat}} = \frac{W_{\mathrm{kin}}}{\tau\langle I^2\rangle} = \frac{2m_e}{3\pi ne^2R\tau} = 2.46\ \Omega$.	1.0

2.9	For equivalent scattering resistance $R_{\mathrm{scat}} = \frac{P_{\mathrm{scat}}}{\langle I^2\rangle}$ and for harmonic oscillations we can average the velocity squared over one period of oscillations, so $\langle v^2\rangle = \frac{1}{2}\omega_{\mathrm{p}}^2x_0^2$. Together it yields $R_{\mathrm{scat}} = \frac{Q^2x_0^2\omega_{\mathrm{p}}^4}{12\pi\varepsilon_0c^3}\frac{16R^2}{9Q^2\langle v^2\rangle} = \frac{8\omega_0^2R^2}{27\pi\varepsilon_0c^3} = 2.45\ \Omega$.	1.0

2.10a	Ohm's law for a *LCR* serious circuit is $I_0 = \frac{V_0}{\sqrt{(R_{heat}+R_{scat})^2+\left(\omega L-\frac{1}{\omega C}\right)^2}}$. At the resonance frequency time-averaged voltage squared is $\langle V^2\rangle = Z_R^2\langle I^2\rangle = (R_{\mathrm{heat}}+R_{\mathrm{scat}})^2\langle I^2\rangle$. And from (2.5b) $\langle V^2\rangle = \frac{1}{2}V_0^2 = \frac{8}{9}R^2E_0^2$, so Ohm's law results in $\langle I^2\rangle = \frac{8R^2E_0^2}{9(R_{\mathrm{heat}}+R_{\mathrm{scat}})^2}$. The time-averaged power losses are $P_{\mathrm{heat}} = R_{\mathrm{heat}}\langle I^2\rangle = \frac{8R_{\mathrm{heat}}R^2}{9(R_{\mathrm{heat}}+R_{\mathrm{scat}})^2}E_0^2$ and $P_{\mathrm{scat}} = \frac{8R_{\mathrm{scat}}R^2}{9(R_{\mathrm{heat}}+R_{\mathrm{scat}})^2}E_0^2 = \frac{R_{\mathrm{scat}}}{R_{\mathrm{heat}}}\langle P_{\mathrm{heat}}\rangle$.	1.2
2.10b	Starting with the electric field amplitude $E_0 = \sqrt{2S/(\varepsilon_0c)} = 27.4\ \mathrm{kV/m}$, we calculate $P_{\mathrm{heat}} = 6.82\ \mathrm{nW}$ and $P_{\mathrm{scat}} = 6.81\ \mathrm{nW}$.	0.3

Steam generation by light

2.11a	Total number of nanoparticles in the vessel: $N_{\mathrm{np}} = h^2a\,n_{\mathrm{np}} = 7.3\times10^{11}$. Then the total time-averaged Joule heating power: $P_{\mathrm{st}} = N_{\mathrm{np}}P_{\mathrm{heat}} = 4.98\ \mathrm{kW}$. This power goes into the steam generation: $P_{\mathrm{st}} = \mu_{\mathrm{st}}L_{\mathrm{tot}}$, with $L_{\mathrm{tot}} = c_{\mathrm{wa}}(T_{100}-T_{\mathrm{wa}}) + L_{\mathrm{wa}} + c_{\mathrm{st}}(T_{\mathrm{st}}-T_{100}) = 2.62\times10^6\ \mathrm{J\ kg^{-1}}$. Thus the mass of steam produced in one second is: $\mu_{\mathrm{st}} = \frac{P_{\mathrm{st}}}{L_{\mathrm{tot}}} = 1.90\times10^{-3}\ \mathrm{kg\ s^{-1}}$.	0.6
2.11b	The power of light incident on the vessel $P_{\mathrm{tot}} = h^2S = 0.01\mathrm{m}^2\times 1\ \mathrm{MW\ m^{-2}} = 10.0\ \mathrm{kW}$, and the power directed for steam production by nanoparticles is given in 2.11a. Efficiency of the process is $\eta = \frac{P_{\mathrm{st}}}{P_{\mathrm{tot}}} = \frac{4.98\ \mathrm{kW}}{10.0\ \mathrm{kW}} = 0.498$.	0.2

	Total	**12.0**